Improved Understanding of Bimodal Coupled Bridge Flutter Based on Closed-Form Solutions

Xinzhong Chen

Abstract: Analysis of an aeroelastic bridge system consisting of the fundamental vertical and torsional modes of vibration offers an expeditious assessment of bridge flutter performance. It also produces valuable insight into the multimode coupled bridge response to strong winds. This paper presents closed-form formulations for estimating the modal frequencies, damping ratios, and coupled motions of the bimodal coupled aeroelastic bridge system at varying wind velocities. The derivation of these formulations is based on the assumption of low-level damping of the aeroelastic bridge system. This assumption has also been adopted in current modeling of self-excited forces and the analysis of coupled flutter through complex eigenvalue analysis. This framework leads to a formula for determining the critical flutter velocity of bridges with generic bluff deck sections, which not only provides an analytical basis for Selberg’s empirical formula, but also serves as its extension to generic bridges. This formula gives a single parameter or index as a function of flutter derivatives to describe the flutter efficiency of a given bridge section, which facilitates comparison of aerodynamic characteristics of different bridge deck sections. The accuracy of the proposed framework is illustrated through long span bridge examples with a variety of structural and aerodynamic characteristics. Based on the proposed framework, the significance of structural and aerodynamic characteristics on the development of coupled motion and the evolution of modal damping is discussed, which helps to better understand how and where bridges may be tailored for better flutter performance. It is pointed out that coupled bridge flutter is initiated from the modal branch that has a higher modal frequency and is characterized by coupled motion in which torsional motion lags vertical motion. The generation of coupled flutter instability is driven by the negative damping effect caused by the coupled self-excited forces.

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Introduction

Flutter instability is one of the major concerns in the design and construction of long span bridges. Since the collapse of the Tacoma Narrows Bridge in 1940, extensive studies on bridge flutter in both experimental and analytical/numerical arenas have been conducted focusing on improved understanding of its generation mechanism, accurate modeling of self-excited aerodynamic forces, better prediction of critical flutter velocity, and efficient strategies for enhancing flutter performance. The development of the spring-mounted bridge section model testing technique in wind tunnels has opened the door for extensive studies on bridge flutter toward seeking stable bridge deck sections. Early pioneering work on bridge flutter analysis was performed by Bleich (1948) in which bimodal coupled bridge flutter consisting of fundamental vertical bending and torsional modes of vibration was addressed using airfoil aerodynamic theory. This analysis framework led to extensive numerical studies concerning the influence of bridge mass and frequency parameters on the critical flutter velocity. These numerical studies provided a basis for the empirical formula suggested by Selberg (1961) for estimating the critical flutter velocity of bridges with a flat plate section. The introduction of flutter derivatives and the development of their identification schemes through wind tunnel testing offer realistic modeling of aerodynamic forces on bluff bridge sections (e.g., Ukeguchi et al. 1966; Sabzevari and Scanlan 1968; Scanlan and Tomko 1971; Scanlan 1978, 1993; Davenport and King 1984; Sarkar et al. 1994; Jakobsen 1995; Matsumoto et al. 1995; Chen and Kareem 2002). In recent years, multimode coupled flutter analysis frameworks have been developed for advanced prediction of coupled bridge flutter, integrating the structural characteristics of multiple modes and the experimentally obtained aerodynamic forces on bridge sections (e.g., Miyata et al. 1994; Jones et al. 1998; Chen et al. 2000a, b; Chen and Kareem 2003a; Diana et al. 1999). The solution of the equations of motion of aeroelastic bridge systems through a complex eigenvalue analysis provides information on how the self-excited forces influence modal frequencies, damping ratios, and intermodal coupling as wind velocity increases (e.g., Chen et al. 2000a).

The participation of structural modes in bridge flutter can be identified through their amplitudes in flutter motion and, more precisely, through their contributions to the flutter modal damping (Chen et al. 2000a). Many analysis examples have shown that bridge flutter is often dominated by the fundamental vertical bending and torsional modes with secondary contributions from other modes (e.g., Chen et al. 2000a); therefore, bimodal coupled flutter analysis that involves only the two fundamental modes remains a sufficiently accurate and useful tool for an expeditious evaluation of bridge flutter performance, especially for seeking...
the best bridge deck sections with superior aerodynamic characteristics at a preliminary design stage. The bimodal coupled system also serves the foundation for the spring-supported section model testing technique that has routinely been utilized in bridge flutter studies. Improved understanding of the bimodal coupled flutter promises to offer insights into bridge flutter involving the coupling of multiple modes.

Bimodal coupled flutter can be numerically predicted through a complex eigenvalue analysis approach. While this approach is computationally effective, it is not straightforward to unveil the significance of structural and aerodynamic characteristics on the generation of coupled flutter without conducting extensive parametric studies. In light of this background, Matsumoto (1999) introduced a step-by-step iterative analysis framework attempting to capture the mechanism surrounding the evolution of bridge flutter. Chen and Kareem (2006) have presented a framework with closed-form formulations to estimate the modal frequencies, damping ratios, and coupled motions of both vertical and torsional modal branches at varying wind velocities. Recently, the writer has noted that Nakamura (1978) suggested a set of formulas giving the frequency and damping ratio and coupled motion between vertical bending and torsion in the torsional modal branch near the critical flutter velocity. These research efforts on the analytical solution of the bimodal coupled flutter are not intended to replace the current numerical frameworks; rather, they focus on developing enhanced tools that would enhance our understanding of the underlying physics of bridge flutter.

This paper addresses the analytical framework for bimodal coupled bridge flutter proposed in Chen and Kareem (2006) with its further simplifications and extensive illustrations concerning its accuracy through a variety of bridge examples. Based on this framework, a formula for determining the critical flutter velocity of bridges with generic bluff deck sections is obtained, which serves as an extension of Selberg’s empirical formula to generic bridges. It provides a simple parameter or index as a function of flutter derivatives to describe the flutter efficiency of a given deck section, which facilitates comparison of aerodynamic characteristics of different bridge deck sections. Based on the proposed framework, the significance of structural and aerodynamic characteristics on the development of coupled motion and the evolution of modal damping is discussed, which helps to better understand how and where bridges may be tailored for better flutter performance.

Closed-Form Solutions

The aeroelastic bridge system consisting of fundamental vertical and torsional modes is considered. In the cases where the fundamental torsional mode is antisymmetric, the corresponding fundamental bending mode is referred to as the fundamental antisymmetric mode. Otherwise, both are referred to as fundamental symmetric modes. Only the aerodynamic forces on the bridge deck, which generally dominate the aerodynamic performance, are included, while the structural characteristics of all bridge elements have been modeled in the quantification of modal properties. The dynamic displacements of the bridge deck in the vertical direction and torsion are given as $h(x,t) = h_1(x)q_1(t)$ and $\alpha(x,t) = \alpha_1(x)q_1(t)$, where $h_1(x)$ and $\alpha_1(x)$ are mode shapes, $q_1(t)$ is a generalized modal coordinate. The self-excited (se) lift (downward) and pitching moment (nose up) acting on the bridge deck section per unit length are given by (Scanlan and Tomko 1971; Scanlan 1978)

$$L_{se}(t) = \frac{1}{2} \rho U^2 (2h) \left( kH_1^1 \frac{h}{U} + kH_2^1 \frac{b\alpha}{U} + k^2 H_3^1 \alpha + k^2 H_4^1 \frac{h}{b} \right)$$

$$M_{se}(t) = \frac{1}{2} \rho U^2 (2h^2) \left( kA_1^1 \frac{h}{U} + kA_2^1 \frac{b\alpha}{U} + k^2 A_3^1 \alpha + k^2 A_4^1 \frac{h}{b} \right)$$

where $\rho =$ air density; $U =$ mean wind velocity; $B =$ bridge deck width; $k =$ $\omega b/U =$ reduced frequency; $\omega =$ circular frequency of motion; and $H_i^j$ and $A_i^j (j=1,2,3,4) =$ flutter derivatives that are functions of reduced frequency.

The governing equations of bridge motion in terms of the generalized modal coordinates excluding the turbulence-induced buffeting forces are expressed as

$$M \ddot{q} + C \dot{q} + K q = \frac{1}{2} \rho U^2 \left( A_1 q + \frac{b}{U} A_2 \dot{q} \right)$$

where $M$, $C$, and $K =$ generalized mass, damping, and stiffness matrices, respectively; $m_j$, $\xi_j$, and $\omega_j (j=1,2) =$ generalized modal masses, damping ratios, and frequencies of vertical and torsional modes; $A_j$ and $A_{j*} =$ aerodynamic stiffness and damping matrices; $G_{ij} = \int \rho r \sigma_r^i(s) \sigma_s^j(s) ds$ (where $r, s = h, \alpha$; $i, j = 1,2 =$ modal integrals).

The presence of the aerodynamic stiffness and damping matrices renders the equations of bridge motion to be frequency dependent and coupled. The modal frequencies, damping ratios, and coupled motions affected by the self-excited forces can be quantified through the solution of the following complex eigenvalue problem by setting $q(t) = q_0 e^{\lambda t}$ in Eq. (3):

$$\left( \lambda^2 M + \lambda C + K \right) q_0 e^{\lambda t} = \frac{1}{2} \rho U^2 (A_1 + \lambda A_2) q_0 e^{\lambda t}$$

where $\lambda = -\xi \omega + i \omega \sqrt{1 - \xi^2}$; $\omega$ and $\xi =$ frequency and damping ratio of the complex modal branch; $\lambda = \lambda b/U$; and $i = \sqrt{-1}$. The critical flutter velocity is determined when one of the modal damping ratios becomes zero, beyond which the system becomes unstable because of negative damping.

The hypothesis of low-level damping is generally implied in the modeling of self-excited forces and conventional bridge flutter analysis. Because of low-level damping, the decaying or growing of structural motion has little influence on the self-excited forces that are only functions of reduced frequency or reduced velocity rather than both the reduced frequency and damping. The flutter derivatives are experimentally quantified for bridge motion with low damping (free-vibration method) or zero damping (forced-vibration method). This hypothesis concerning system damping has also routinely been made in the flutter analysis schemes, such as the so-called $p-k$ and $p$ methods and the conventional com-
plex eigenvalue analysis. On the basis of this hypothesis, by setting \( \mathbf{A}_s + \mathbf{\lambda} \mathbf{A}_g = \mathbf{A}_s + (ik) \mathbf{A}_g \) and eliminating the terms involving the higher order of the damping ratio, Eq. (5) becomes (Chen and Kareem 2006)

\[
[- \omega^2 + 2i \xi_1 \bar{\omega}_1 \omega + \bar{\omega}_1^2]q_{10e}^{\omega} = \mu D \omega^3 (H_3 + iH_2) \\
\times (G_{\alpha z_{z_2}} / G_{h_4 h_5})^{1/2} (bq_{20}) e^{\omega t} \tag{6}
\]

\[
[- \omega^2 + 2i \xi_2 \bar{\omega}_2 \omega + \bar{\omega}_2^2] (bq_{20}) e^{\omega t} = \nu D \omega^2 (A_3 + iA_4) \\
\times (G_{h_4 h_5} / G_{\alpha z_{z_2}})^{1/2} q_{10e}^{\omega} \tag{7}
\]

where \( \bar{\omega}_1 \) and \( \bar{\omega}_2 (j = 1, 2) \) are frequencies and damping ratios that are influenced only by the uncoupled self-excited forces, i.e., the lift caused by vertical motion and the pitching moment caused by torsion, associated with \( H_3, H_4, A_3, \) and \( A_4, \) which are defined at the reduced frequency \( k = \omega b / U \)

\[
\bar{\omega}_1 = \omega_1^* [1 - \mu (\omega / \omega_1^*)^2 H_3^2]^{1/2} \tag{8}
\]

\[
\bar{\omega}_2 = \xi_1 \bar{\omega}_1 \omega / \omega_1^* - 0.5 \mu (\omega / \omega_1^*) \xi_1 \omega / \omega_1^* \tag{9}
\]

\[
\bar{\omega}_2 = \omega_2^* [1 - \nu (\omega / \omega_2^*)^2 A_3^2]^{1/2} \tag{10}
\]

\[
\bar{\omega}_2 = \omega_2^* (\omega / \omega_2^*)^2 - 0.5 \nu (\omega / \omega_2^*) A_3^2 - \xi_2 \omega / \omega_1^* \tag{11}
\]

\[
\nu = \mu b^2 / m, \quad \nu = \mu b^2 / H_3, \quad m = m_1 / G_{h_4 h_5}, \quad \text{and} \quad D = G_{h_4 h_5} / G_{\alpha z_{z_2}}^{1/2} \text{similarity factor in modal shapes of the fundamental vertical and torsional modes, i.e., } D = 0 \text{ and } 1 \text{ indicate that two mode shapes are orthogonal and identical, respectively.}
\]

Accordingly, the modal frequency and damping ratio of the torsional modal branch, i.e., \( \omega = \omega_1, \) and \( \xi = \xi_1, \) the associated amplitude ratio, \( \Phi, \) and phase difference between the vertical motion and torsion, \( \psi, \) as defined by \( bq_{20} / q_{10} = \Phi e^{i \psi}, \) i.e., a positive value of \( \psi \) indicates torsional motion lags vertical motion, are expressed as

\[
\omega_1 = \omega_1^* (1 + \mu H_3^2 + \mu v D^2 \Phi ' \cos \psi ')^{-1/2} \tag{12}
\]

\[
\xi_1 = \xi_1^* \omega_1 / \omega_1^* - 0.5 \mu \omega_1^* \xi_1 \omega_1 / \omega_1^* \sin \psi ' \tag{13}
\]

\[
\Phi = v D R_{d1} [[(A_3^2 + (A_4^2) / G_{h_4 h_5} / G_{\alpha z_{z_2}}^{1/2}]]^{1/2} \tag{14}
\]

\[
\phi = \tan^{-1}(A_3^2 / A_4^2) + \tan^{-1}[2 \xi_2 (\omega / \omega_2^*) / [1 - (\omega / \omega_2^*)^2]] \tag{15}
\]

where

\[
R_{d1} = (\omega / \omega_2^*)^2 [1 - (\omega / \omega_2^*)^2] + [2 \xi_2 (\omega / \omega_2^*) / [1 - (\omega / \omega_2^*)^2]]^{-1/2} \tag{16}
\]

\[
\Phi ' = R_{d1} [(H_3^2 + (H_4^2) / (A_3^2 + (A_4^2)]^{1/2} \tag{17}
\]

\[
\phi ' = \tan^{-1}(H_3^2 / H_4^2) + \phi \tag{18}
\]

Similarly, for the torsional modal branch, the frequency and damping ratio, i.e., \( \omega = \omega_2, \) and \( \xi = \xi_2, \) and the amplitude ratio, \( \Psi ', \) and phase difference between the vertical motion and torsion, \( \psi ', \) as defined by \( bq_{20} / q_{10} = \Psi e^{i \psi '}, \) i.e., a positive value of \( \psi \) indicates torsional motion lags vertical motion, are given by

\[
\omega_2 = \omega_2^* (1 + \nu A_3^2 + \mu v D^2 \Psi ' \cos \psi ')^{-1/2} \tag{19}
\]

\[
\xi_2 = \xi_2^* \omega_2 / \omega_2^* - 0.5 \nu A_3^2 - 0.5 \mu v D^2 \Psi ' \sin \psi ' \tag{20}
\]

\[
\Psi = \mu D R_{d2} [[(H_3^2 + (H_4^2) / (G_{\alpha z_{z_2}} G_{h_4 h_5})]^{1/2} \tag{21}
\]

\[
\psi = \tan^{-1}(H_3^2 / H_4^2) - \tan^{-1}[2 \xi_2 (\omega / \omega_2^*) / [1 - (\omega / \omega_2^*)^2]] \tag{22}
\]

where

\[
R_{d2} = (\omega_2 / \omega_2^*)^2 [1 - (\omega_2 / \omega_2^*)^2] + [2 \xi_2 (\omega_2 / \omega_2^*) / [1 - (\omega_2 / \omega_2^*)^2]]^{-1/2} \tag{23}
\]

\[
\Psi ' = R_{d2} [(H_3^2 + (H_4^2) / (A_3^2 + (A_4^2)]^{1/2} \tag{24}
\]

\[
\psi ' = \tan^{-1}(A_3^2 / A_4^2) + \psi \tag{25}
\]

It is clear that quantification of the modal frequencies and damping ratios of the coupled system requires iterative calculations, similar to the conventional complex eigenvalue analysis approach, as the terms on the right side of the formulations involve unknown frequency and damping. However, these iterative calculations converge rapidly and can even be eliminated by inserting respective values of frequencies and damping ratios at the immediately previous wind velocity. It is worthy of mention that for a low-level damping, particularly for the torsional modal branch around the critical flutter velocity, the iterative calculations attributed to unknown modal damping may be eliminated by simply ignoring its influence. Furthermore, except in the case where the two modal frequencies are close together, in which the influence of coupled forces tends to separate both frequencies, the modal frequencies are generally less affected by the coupled self-excited forces. Subsequently, the frequencies of the coupled system can often be approximated by those of the corresponding uncoupled system.

When the coupled self-excited forces, i.e., the lift caused by torsion and the pitching moment caused by vertical motion associated with \( H_3, H_4, A_3, A_4, \) are negligibly small, the equations of motion become uncoupled. For this uncoupled system, the modal frequency and damping ratio of each modal branch are affected only by the uncoupled self-excited forces and can be estimated separately as (Scanlan 1978)

\[
\omega_{10} = \omega_1^* (1 + \mu H_3^2)^{-1/2} \tag{26}
\]

\[
\xi_{10} = \xi_1^* \omega_1 / \omega_1^* - 0.5 \mu H_3^2 \tag{27}
\]

\[
\omega_{20} = \omega_2^* (1 + \nu A_3^2)^{-1/2} \tag{28}
\]

\[
\xi_{20} = \xi_2^* \omega_2 / \omega_2^* - 0.5 \nu A_3^2 \tag{29}
\]

It is noted that when the value of wind velocity is prescribed, quantification of each modal frequency based on Eq. (26) or (28) requires an iterative calculation because the flutter derivatives are functions of reduced frequency. However, when the analysis is conducted for a prescribed reduced frequency or reduced wind velocity, this iterative calculation can be circumvented. The modal damping ratios are quantified following the modal frequencies. It is obvious that the single-vertical-mode flutter, i.e., galloping, or single-torsional-mode flutter, i.e., torsional flutter, may be developed only when \( H_3^2 > 0 \) or \( A_3^2 > 0, \) respectively. For bridges with relatively slender sections, as \( H_3^2 < 0 \) and \( A_3^2 < 0, \) the uncoupled self-excited forces result in positive aerodynamic damping. Accordingly, only a coupled flutter associated with coupled vertical and torsional motions may be developed as the result of combined action of both uncoupled and coupled self-excited forces.
Simplified Closed-Form Solutions and Critical Flutter Velocity

In general, for the bimodal coupled system, \( \tilde{\omega}_1 \neq \omega_{10} \) and \( \tilde{\omega}_2 \neq \omega_{20} \) because the flutter derivatives involved in these formulations correspond to distinct reduced frequencies. However, the modal frequencies of the coupled system are generally very close to those of the corresponding uncoupled system at the same wind velocity. In addition, the influence of the uncoupled self-excited forces on modal frequencies is not sensitive to the change in reduced frequency. Therefore, \( \omega_1 = \tilde{\omega}_1 = \omega_{10} \left[ 1 - \mu \left( \tilde{\omega}_1 / \omega_{10} \right)^2 H_1^2 \right]^{1/2} = \omega_{10} \) and \( \omega_2 = \tilde{\omega}_2 = \omega_{20} \left[ 1 - \nu \left( \tilde{\omega}_2 / \omega_{20} \right)^2 A_2^2 \right]^{1/2} = \omega_{20} \).

When the modal frequencies of the uncoupled system are well separated, \( R_{d1} \) and \( R_{d2} \) are insensitive to the values of \( \xi_2 \) and \( \xi_1 \) and thus may be approximated as \( R_{d1} \approx |1 - (\omega_{10} / \omega_{10})^2| \) and \( R_{d2} \approx |1 - (\omega_{20} / \omega_{20})^2| \). Moreover, the phase angles \( \theta_{d1} = \tan^{-1} \left( \frac{2 \xi_1 (\omega_1 / \omega_2) / \omega_{10} \right) \right] \) and \( \theta_{d2} = \tan^{-1} \left[ \frac{2 \xi_2 (\omega_2 / \omega_2) / \omega_{20} \right) \right] \) are close to 0 and \( \pi \), respectively. In addition, \( H_1 A_1 + H_2 A_2 = H_1 A_1 \) is generally acceptable, as will be illustrated later. Subsequently, the modal damping ratios given by Eqs. (13) and (20) can be simplified as

\[
\xi_1 = \xi_1^* \left( \omega_1 / \omega_{10} \right) - 0.5 \mu H_1 + 0.5 \mu \nu D_1 \frac{H_1^2}{A_1^2} \left[ 1 - (\omega_{10} / \omega_{10})^2 \right] \quad (30)
\]

\[
\xi_2 = \xi_2^* \left( \omega_2 / \omega_{20} \right) - 0.5 \nu A_2 + 0.5 \nu \mu D_2 \frac{H_2^2}{A_2^2} \left[ 1 - (\omega_{20} / \omega_{20})^2 \right] \quad (31)
\]

The critical flutter velocity is determined at the condition of zero modal damping. By setting \( \omega_{10} = \omega_1 \) and \( \omega_{20} = \omega_2 \), Eq. (31) with \( \xi_2 = 0 \) gives

\[
k^2 - \eta^2 k^2 (1 + \nu A_2)^2 - \mu D_2 (-k^2 H_2^2) (k A_2)^2 \]

\[
f\left[ (-k^2 H_2^2) (k A_2)^2 \right] = 0 \quad (32)
\]

where \( \eta = \omega_1 / \omega_2 \) = frequency ratio; \( k = \omega_0 b / U_{cr} \); and \( U_{cr} \), critical flutter velocity.

By noting that \( k_0^2 = k^2 (1 + \nu A_2) \), where \( k_0 = \omega_0 b / U_{cr} \), the preceding equation becomes

\[
k_0^2 (1 - \eta^2) = \nu (k^2 A_2) + \mu D_2 (-k^2 H_2^2) (k A_2)^2 \]

\[
f\left[ (-k^2 H_2^2) (k A_2)^2 \right] = 0 \quad (33)
\]

which leads to

\[
U_{cr} = \gamma \omega_0 b \sqrt{1 - \eta^2 \left( \frac{m r}{p b^2} \right)} \quad (34)
\]

\[
\gamma = 1 / \sqrt{F_1 + F_2} \quad (35)
\]

\[
F_1 = (r / b) D_1 \left( -k^2 H_1^2 \right) (k A_1) \left[ (-k^2 A_1^2 + 2 k \xi_2 (1 + \nu A_1)^2 / \nu) \right] \quad (36)
\]

\[
F_2 = (b / r) (k^2 A_2) \quad (37)
\]

The intersection of the curve by Eq. (34) as a function of reduced frequency \( k \) with the curve by \( U_{cr} = \omega_0 b / k_0 = 1 + 1/2 \omega_{10} / \omega_{10} b / k \) provides an alternative format for estimating the critical flutter velocity, which results in the same result as Eq. (31).

**Illustration of the Accuracy of Proposed Framework**

When \( F_1 \) and \( F_2 \) are not distinctly different, the arithmetic mean \( (F_1 + F_2) / 2 \) can be approximated by the geometrical mean \( \sqrt{F_1 F_2} \). The parameter \( \gamma \) can then be estimated as

\[
\gamma = \left[ \left( -k^2 H_2^2 (k A_2)^2 \right) \right]^{1/2} / \sqrt{2D} \quad (38)
\]

In the case of a spring-mounted bridge section model with two-degree-of-freedom (2DOF) motions in vertical and torsional directions, one has \( G_{11} = G_{22} = G_{12} = 1 \) and \( D = 1 \). For a flat plate section, the flutter derivatives can be quantified through the Theodorsen function. Fig. 1 shows the corresponding parameter \( \gamma \) as a function of reduced wind velocity estimated by Eq. (35) with \( r / b = 0.6, 0.8 \), and 1.0 for typical bridge sections and by Eq. (38), in which the structural damping is neglected. At the range of reduced wind velocity \( U / B = 10-30 \), the parameter \( \gamma \) is between 0.405 and 0.432, which is very close to the value \( \gamma = 0.416 \) as suggested in Selberg’s empirical formula with an error of less than 5%.

Selberg’s formula was developed based on the numerical predictions of critical flutter velocities of bridges with a flat plate section and variety of mass and frequency parameters. While this empirical formula proves to be very accurate, according to the writer’s limited knowledge, there has been no analytical basis in support of this formula concerning the influence of mass and frequency parameters on flutter velocity. In this context, Eq. (34) may be regarded as an analytical basis for Selberg’s formula. More importantly, Eq. (34) serves as its extension to bridges with generic bluff deck sections. It clearly points to the significance of structural and aerodynamic characteristics on bridge flutter performance, which aids in better understanding of how and where the structure may be tailored for better flutter performance. For example, flutter performance may be improved by increasing the ratio of torsional modal frequency to vertical modal frequency, by increasing the structural effective mass and torsional frequency, and by decreasing the similarity factor of the vertical and torsional mode shapes. On the other hand, by tailoring the geometric shape of the bridge deck section, the aerodynamic characteristics may be modified to achieve an increased absolute value of \( A_2^* \) and reduced absolute values of \( H_2^*, A_1^*, \) and \( A_3^* \), so that parameter \( \gamma \) can be increased leading to enhanced bridge flutter performance.

**Long span suspension bridges with a center span of nearly 2,000 m with a variety of deck sections are considered as examples to illustrate the accuracy of the proposed framework. The bridge deck sections include rectangular sections with width to**
depth ratios $B/D=12.5, 15, \text{ and } 20$ (Matsumoto 1995), a streamlined box section with $B/D=4.45$ (Jakobsen 1995), and a flat plate. Some of their flutter derivatives are shown in Fig. 2. The bridge deck width is $B=35.5 \text{ m}$. The nondimensional mass and polar moment of inertia parameters are $\mu=0.0088$ and $\nu=0.0109$. The fundamental vertical and torsional modes are symmetric modes with a frequency ratio of $2.41$. The similarity factor of these two mode shapes is $D=0.9637$, which is very close to unity, indicating a high similarity.

Fig. 3 shows the comparison of the modal frequencies and damping ratios at varying wind velocities predicted using the conventional framework based on numerical eigenvalue analysis and the proposed framework with analytical closed-form solutions in the case of a flat plate section. The structural modal damping is neglected in order to focus the discussion only on the influence of self-excited forces. For this section, consideration of modal damping only slightly changes the critical flutter velocity. The modal frequencies and damping ratios of the corresponding uncoupled system are also provided in Fig. 3. Fig. 4 shows the results in the case of the box section. Fig. 5 compares the modal damping ratios at varying wind velocities predicted by using the

Fig. 2. Comparison of flutter derivatives for different deck sections: (a) $H_3^*; (b) A_1^*; (c) A_2^*$; and (d) $A_3^*$

Fig. 3. Comparison of predicted modal frequencies and damping ratios (flat plate section): (a) frequency versus wind velocity; (b) damping ratio versus wind velocity

Fig. 4. Comparison of predicted modal frequencies and damping ratios (box section): (a) frequency versus wind velocity; (b) damping ratio versus wind velocity

Fig. 5. Accuracy of the simplified closed-form solution in predicting the modal damping ratios: (a) flat plate section; (b) box section
simplified closed-form solutions for both flat plate and box deck sections [Eqs. (30) and (31)]. Table 1 gives the comparison of the critical flutter velocities for different deck sections predicted using different schemes. The proposed framework performs very well in all cases, which demonstrates its accuracy and effectiveness. There are minor differences in the predicted damping of the vertical bending modal branch at higher wind velocities where the assumption of low damping breaks down. The critical flutter velocity predicted by using the simplified closed-form solution or Eq. (34) are slightly larger, but the error is within the range of less than 5%. In the case of the flat plate, Selberg’s formula gives a critical flutter velocity of 67.7 m/s.

Table 2 shows the influence of flutter derivatives on the critical flutter velocity in the case of the box section. It is obtained by altering individual flutter derivatives independently in the analysis. Such an analysis may be less practical as the potential interrelationship among flutter derivatives may not permit independent change in individual flutter derivative. Nonetheless, it provides information concerning the sensitivity of the critical flutter velocity to each individual flutter derivative. As previously pointed out in the literature (e.g., Matsumoto 1995), the flutter derivatives \( H_{31}, A_{31}, A_{32} \), and \( A_{31} \) are the most important ones that dominate the flutter performance. The dominant role of these flutter derivatives can also be clearly identified through the formulation of the parameter \( \gamma \), which is the function of only these important flutter derivatives. The results show that in all cases the proposed scheme performs very well, which again confirms its accuracy.

The accuracy of the proposed framework is further illustrated using 2DOF spring-mounted section models with a variety of structural properties for both flat plate and box sections. In the case of 2DOF section models, \( G_{h1} = G_{h2} = G_{h3} = 1 \) and \( D = 1 \). From Eqs. (6) and (7), it can be seen that the critical reduced flutter velocity, i.e., \( U_{c}/f_{h} \), depends on the nondimensional structural parameters \( \mu, v = \mu/(r/b)^2, \omega_{2}/\omega_{3}, \xi_{1}, \) and \( \xi_{2} \) and the aerodynamic derivatives. Fig. 6 shows the comparison of the critical reduced flutter velocities predicted using different schemes for \( \mu = 0.005, 0.01, \) and 0.02, \( r/b = 0.6 \) and 0.8, and \( \omega_{2}/\omega_{3} \) ranging from 1.2 to 3.0 in the case of flat plate with zero structural modal damping. The selected mass and frequency parameters cover typical long span bridges. The results given by Selberg’s formula are also provided. The proposed closed-form solution gives the same results as the conventional complex eigenvalue analysis. The simplified closed-form solution by Eq. (31) and (34) performs well in general with an error of less than 5%, but with a larger error for bridges with larger mass parameter \( \mu \) (light weight) when the frequency ratio closes to unity. A similar observation is also found in the predictions by Selberg’s formula. This is attributed to the assumption of well-separated frequencies that has been the basis of the simplified formulation and Selberg’s formula. It is seen from Eq. (31) that the influence of this assumption decreases with decreasing mass parameter \( \mu \). There is a slight difference between the simplified formulation and Selberg’s formula. In the simplified solution, the parameter \( \gamma \) depends on the reduced flutter velocity, while in the case of Selberg’s formula, it is taken as a constant of 0.416. In the case of flat plate section, since the parameter \( \gamma \) is insensitive to the reduced frequency, the simplified solution performs very similar to Selberg’s formula. Fig. 7 shows the results in the case of the box section. These results again demonstrate the accuracy of the proposed schemes. As expected, the critical flutter velocity increases with decreasing parameter \( \mu \), increasing parameter \( r/b \), and frequency ratio \( \omega_{2}/\omega_{3} \).

### Generation Mechanism of Coupled Flutter

The proposed framework helps to better understand the development of coupled motions of an aeroelastic bridge system as a result of the self-excited forces. The generation of the coupled vertical motion in the torsional modal branch with a frequency \( \omega_{3} \) can be interpreted as follows. The torsional motion causes aerodynamic lift force with an amplitude proportional to \( \omega_{2}/(H_{31}^2 + H_{32}^2) \), which leads the torsional displacement by a phase angle \( \tan^{-1}(H_{31}^2/H_{32}^2) \). This lift force, as an excitation input of a single-degree-of-freedom system having a frequency \( \bar{\omega}_{1} \) and damping ratio \( \bar{\xi}_{1} \), generates an output of vertical motion. This vertical motion lags the lift force by a phase angle \( \bar{\theta}_{32} \), which is a function of frequency ratio \( \omega_{2}/\bar{\omega}_{1} \) and damping ratio \( \bar{\xi}_{2} \). As \( \omega_{2}/\omega_{1} > 1 \), the angle \( \bar{\theta}_{2} \) is between \( \pi/2 \) and \( \pi \), and is very close to \( \pi \) in the cases with well-separated modal frequencies and a low level of damping. Similar discussions can be made concerning the vertical modal branch, which is characterized by phase angles \( \tan^{-1}(A_{31}^2/A_{32}^2) \) and \( \theta_{31} \). The angle \( \theta_{31} \) is ranging from 0 to \( \pi/2 \) as \( \omega_{1}/\omega_{2} \) and approaches 0 for the system with well-separated modal frequencies and a low level of damping.

Fig. 8 shows the angles \( \tan^{-1}(H_{31}^2/H_{32}^2) \), \( \tan^{-1}(A_{31}^2/A_{32}^2) \) and their summation as functions of reduced wind velocity for those deck sections. Fig. 9 shows the phase differences of the coupled motions in both branches for the bridge examples. It is noted that the
phase difference between vertical and torsional motions in the vertical modal branch, \( \phi = \tan^{-1}\left(\frac{A_1/A_4}{A_1/A_4}\right) - \theta_{d1} \), is generally between 0 and \( \pi \), indicating that the torsional motion leads the vertical motion. On the other hand, in the torsional modal branch, \( \psi = \tan^{-1}\left(\frac{H_2/H_3}{H_2/H_3}\right) - \theta_{d2} \) ranges from 0 to \( \pi/2 \), indicating that the torsional motion lags the vertical motion.

The proposed framework also clearly unveils the role of different aerodynamic force components on modal damping. The uncoupled aerodynamic forces result in positive damping, but the contributions of the coupled forces to both branches depend on the arithmetic signs of \( \sin \phi' \) and \( \sin \psi' \), where \( \phi' = \tan^{-1}\left(\frac{H_2/H_3}{H_2/H_3}\right) + \tan^{-1}\left(\frac{A_1/A_4}{A_1/A_4}\right) - \theta_{d1} \) and \( \psi' = \tan^{-1}\left(\frac{H_2/H_3}{H_2/H_3}\right) + \tan^{-1}\left(\frac{A_1/A_4}{A_1/A_4}\right) - \theta_{d2} \). From previous discussions concerning the values of these angles, it is noted \( \sin \phi' \) is generally negatively valued, but \( \sin \psi' \) is positive valued. Therefore, the coupled self-excited forces produce a positive damping to the vertical modal branch, but a negative damping to the torsional modal branch as shown in Fig. 10. From Eqs. (13) and (20), it is obvious that the level of this negative damping effect depends on the amplitudes of coupled self-excited forces, the nondimensional mass, and polar moment of inertia parameters, frequency ratio, and the similarity factor of fundamental mode shapes. When the negative aerodynamic damping exceeds the positive aerodynamic and structural damping, the bridge becomes negatively damped, which leads to the occurrence of flutter instability.

The role of coupled self-excited forces on modal damping can also be readily clarified from the simplified formulations given by Eqs. (30) and (31). As shown in Fig. 11, \( (H_3 A_1 + H_2 A_4) \) is domi-
For each deck section, the intersection of curves by damping to the vertical modal branch since accordingly, the coupled self-excited forces produce a positive torsional motions generally reduces the critical flutter velocity nated by the value of $H_1^*A$ and is generally negative valued. Accordingly, the coupled self-excited forces produce a positive damping to the vertical modal branch since $\omega_{20}/\omega_{10}>1$ and a negative damping to the torsional modal branch since $\omega_{10}/\omega_{20} < 1$. It is interesting to mention that the coupling of vertical and torsional motions generally reduces the critical flutter velocity even in the case of torsional flutter (Chen and Kareem 2006).

Fig. 10 shows the critical flutter velocities estimated based on Eqs. (34) and (38) as functions of reduced velocity for different deck sections. For each deck section, the intersection of curves by Eq. (34) and $k_0^2=k_0^2(1+\nu A_2)$ determines the critical flutter velocity and corresponding reduced wind velocity. For the instance of the box section, it leads to a critical flutter velocity of 71.5 m/s and reduced velocity of 16.51 as already reported previously in Table 1 and indicated in Fig. 12 by the numbers in the parentheses. As shown in Fig. 13, it is noted that the parameter $\gamma$ given by Eq. (38), a function of reduced wind velocity, is not sensitive to the change in reduced wind velocity when the reduced wind velocity is high. This is as the result of the fact that the flutter derivatives approach their values given by the quasisteady theory with the increasing wind velocity, i.e., $-k_0^2H_1^*\to C_{L1}^*, kA_1^*\to 2C_{L2}^*, -kA_2^*\to 2b\nu C_{M}^*, k^2A_3^*\to 2C_{M2}^*$, where $C_{L1}^*$ and $C_{M2}^*$ are derivatives of static lift and pitching moment coefficients with respect to the angle of incidence. $b_0$ is a constant, which is related to the contribution of velocity of torsional motion to the effective angle of incidence and may not be quantified based on the quasisteady theory (Chen and Kareem 2002). This feature makes Eq. (34) very attractive for an expeditious evaluation of flutter performance of a given deck section, based on their flutter derivatives but without the need of implementing complete procedure of flutter analysis. This formula can be regarded as an extension of Selberg’s from a flat plate section to generic blunt deck sections.

Even though it is understood that flutter performance of a given bridge deck section is dominated by some of its important flutter derivatives, i.e., $H_1^*, A_1^*, A_2^*$, and $A_3^*$, it is often difficult to make a direct comparison of the flutter characteristics of different deck sections purely based on these flutter derivatives. In this context, the parameter $\gamma$ can serve as a useful index for delineating the combined effect of flutter derivatives on flutter. A larger value of the parameter $\gamma$ indicates a better flutter performance of the deck section. As shown in Fig. 12, the box section exhibits the best performance among these sections when the reduced flutter velocity is larger than 11, while bluffer rectangular sections correspond to inferior flutter behaviors. The parameter $\gamma$ directly gives the “flutter efficiency” of the bridge deck section, i.e., the ratio of the critical flutter velocity to that of a flat plate having the same mass and frequency parameters (Walshe and Wyatt 1992). The parameter $\gamma$ as the function of only aerodynamic characteristics facilitates the comparison of aerodynamic performance of
different bridge deck sections over that by using the critical wind velocity, which is influenced by both aerodynamic and structural characteristics.

Equation (34) also points to the influence of structural damping on flutter. Fig. 14 shows the values given by Eqs. (34) and (38) for the rectangular section with $B/D=15$ and the box section at different levels of structural modal damping. As already pointed out in Chen and Kareem (2003b), among others, the effect of modal damping on flutter depends on structural and aerodynamic characteristics, specifically, on the evolution of modal damping with increasing wind velocity. For a hard-type flutter characterized by a negative damping that grows rapidly with increasing wind velocity beyond the onset of flutter, the influence of structural damping on the critical flutter velocity will be insignificant. On the other hand, structural damping may have a notable influence on the critical flutter velocity for a soft-type flutter characterized by a negative damping that grows slowly with increasing wind velocity beyond the onset of flutter.

Concluding Remarks

The proposed framework with closed-form formulations was capable of accurately estimating the bimodal coupled bridge flutter as compared to the conventional complex eigenvalue analysis approach. This framework provided an analytical basis for Selberg’s empirical formula and, more importantly, offered a general formula for estimating critical flutter velocity of bridges with generic bluff deck sections. It led to a single parameter or index, as a function of dominant flutter derivatives, to describe the flutter efficiency of a given bridge deck section. That permits an expeditious comparison of aerodynamic characteristics of different bridge deck sections based on their flutter derivatives but without the need of conducting further analysis. Such an index could be very useful at the preliminary bridge design stage for tailoring bridge deck sections with superior aerodynamic performance. The proposed framework explicitly pointed to the significance of structural and aerodynamic characteristics on the development of coupled motion and evolution of modal damping, which help to better understand how and where structures may be tailored for better flutter performance. It was emphasized that the coupled flutter modal branch was associated with a higher frequency and a coupled motion in which torsional motion lags vertical motion. This type of motion allows the coupled self-excited forces to generate a negative damping effect and eventually lead the system to flutter instability at higher wind velocities.

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References


Fig. 14. Influence of structural damping on critical flutter velocity


